Mechanical Behaviour of Materials

Chapter 02

Elastic behavior

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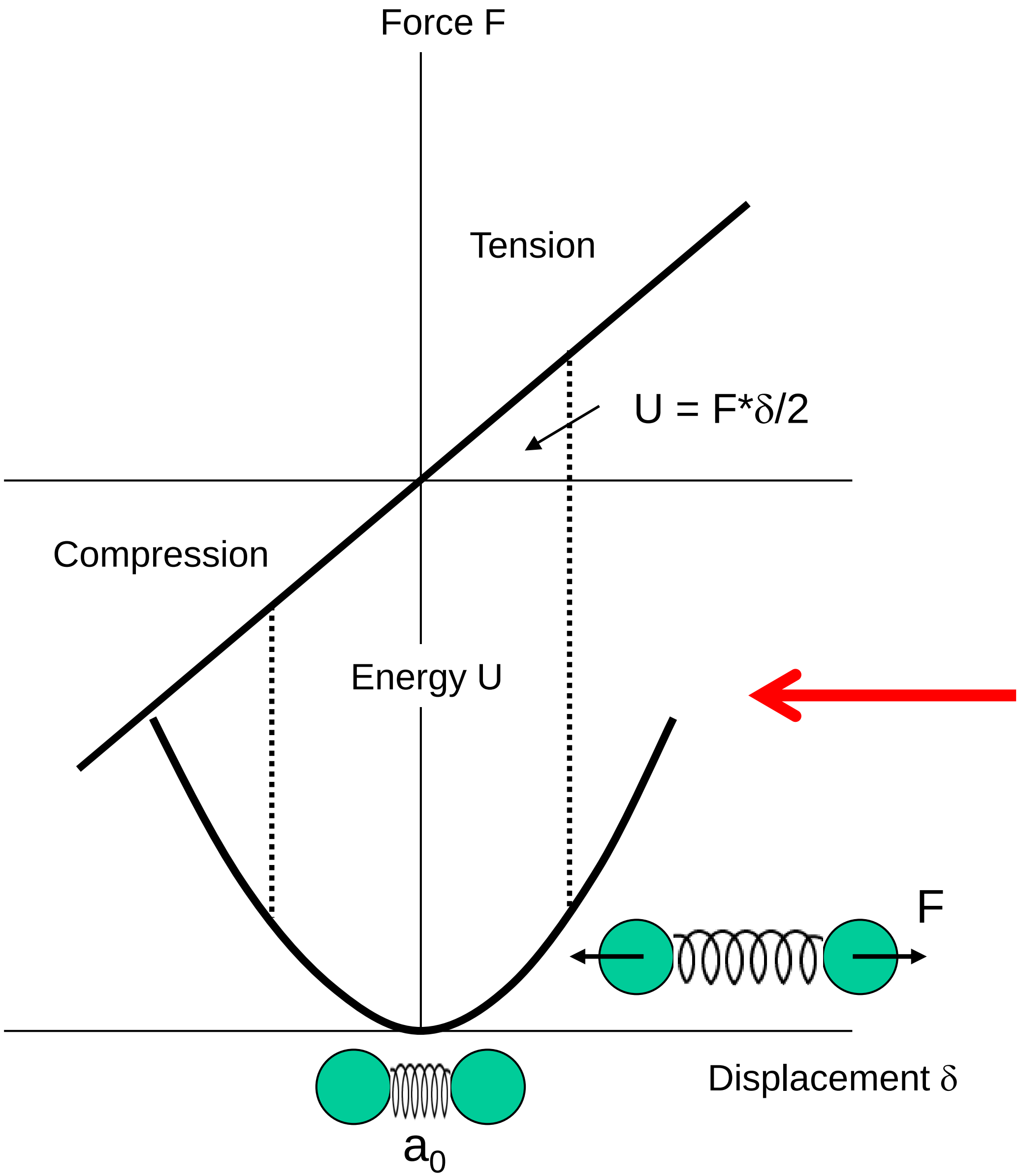
Elasticity

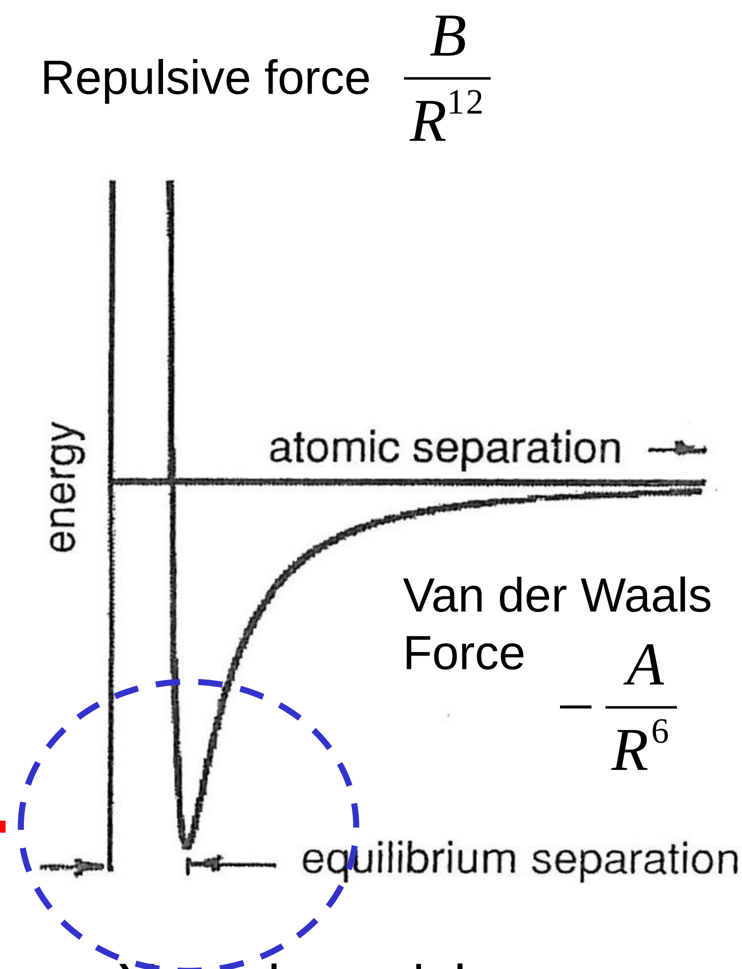
(I) General Hook’s Law

(II) Stiffness/ Compliance matrix

(III) Relation between Young’s modulus and Orientation

Young's modulus





- Young's modulus
  - slope of well
- Melting temperature
  - depth of well

Estimation of the elastic constant

$$E = \frac{d\sigma}{d\varepsilon}\bigg|_{\varepsilon=0} = r_0^{-1}\frac{d^2U}{dr^2}\bigg|_{r=r_0} = \frac{Am(n-m)}{r_0^{m+3}}$$

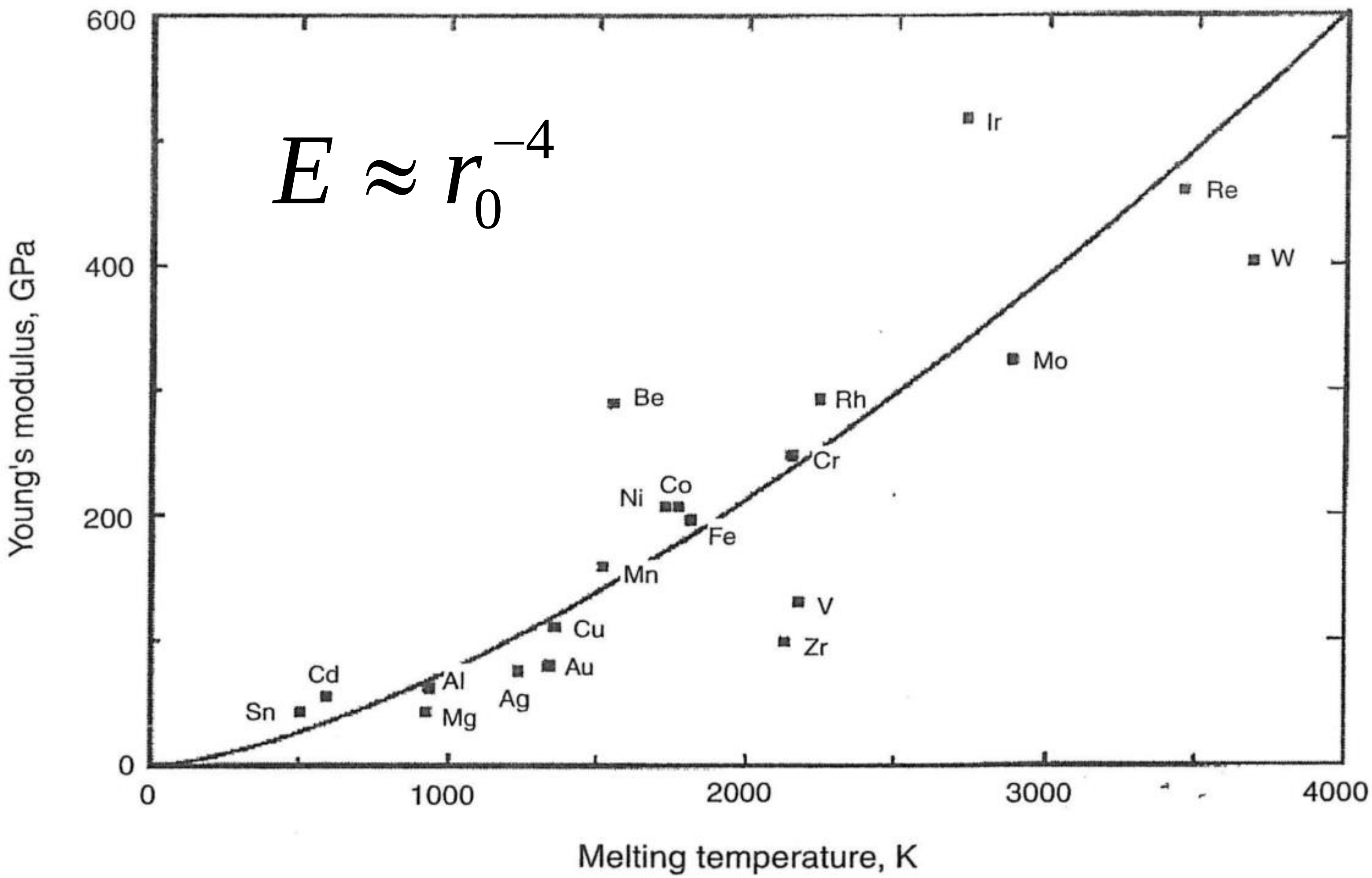
The attraction forces in ionic solids are of a coulombic nature and the exponent is m=1.

$$E = \frac{A(n-1)}{r_0^4} \qquad A = \frac{e^2}{4\pi\varepsilon_0}$$

$$E \approx r_0^{-4}$$

The force is equal to zero at the bottom of the interaction energy curve, which corresponds to the equilibrium separation, r_0.

Young's modulus and melting point



G and E

$$G = \frac{E}{2(1+\upsilon)}$$

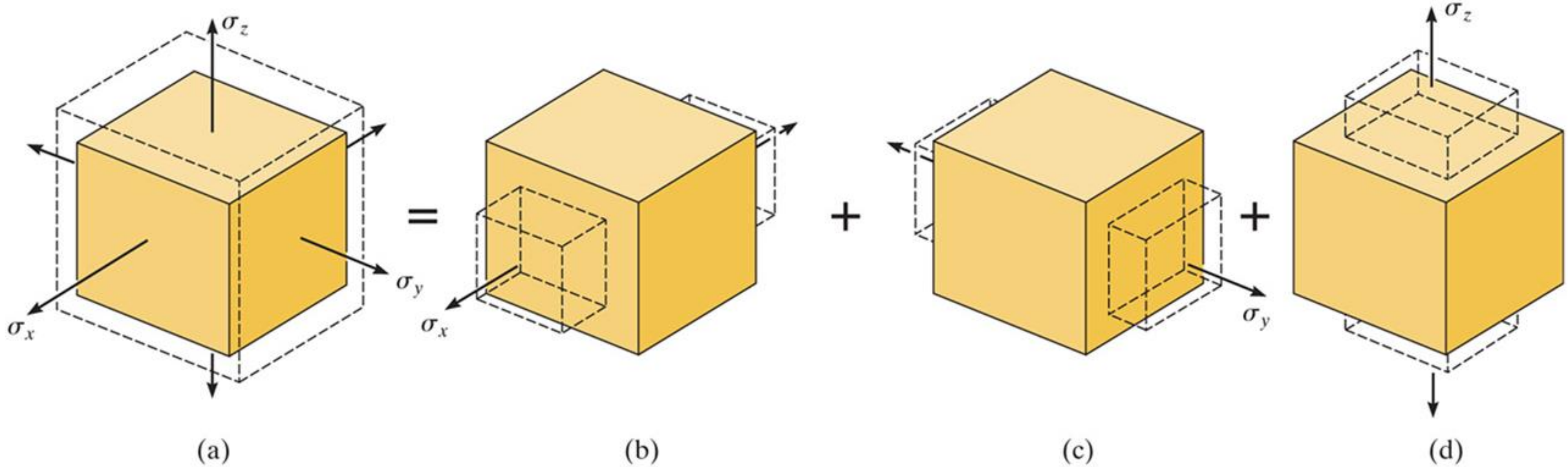
TABLE 1.2 Approximate Elastic Properties of Various Materials (and Dilute Alloys) at Room Temperature

Material	Young's modulus, E (GPa)	Shear modulus, μ (GPa)	Poisson's ratio, ν
Aluminum and aluminum alloys	69–72	24–26	0.35
Copper and copper alloys	125–135	47–50	0.34
Irons and steels	205–215	80–84	0.29
Stainless steels	190–200	75–78	0.33
Titanium and titanium alloys	115	42–44	0.32
Aluminum oxide	380–390	155–165	0.25
Silicon carbide	440–460	195–200	0.14
Glass	70–90	28–32	0.27
Polyethylene (PE)	0.2–2	*	0.4
Polymethylmethacrylate (PMMA)	2–3	*	0.4
Polystyrene (PS)	2–4	*	0.35
Bone**	5–30	3–8	0.25–0.5
Tendon**	0.8–1.5	—	—

*These values are typically about one-third of Young's modulus.

**Bone and tendon are not only strongly anisotropic, but the stiffness also varies with type, position, and moisture content.

3D Hook’s Law



$$\varepsilon'_x = \frac{\sigma_x}{E} \qquad \varepsilon'_x = \frac{-\nu\sigma_y}{E} \qquad \varepsilon'_x = -\frac{\nu\sigma_z}{E}$$

Isotropic elasticity

$$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22} + \sigma_{33}\right)\right]$$

$$\varepsilon_{22} = \frac{1}{E}\left[\sigma_{22} - \nu\left(\sigma_{11} + \sigma_{33}\right)\right]$$

$$\varepsilon_{33} = \frac{1}{E}\left[\sigma_{33} - \nu\left(\sigma_{11} + \sigma_{22}\right)\right]$$

General Hook's Law: Elasticity tensor

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl}$$ S: compliance

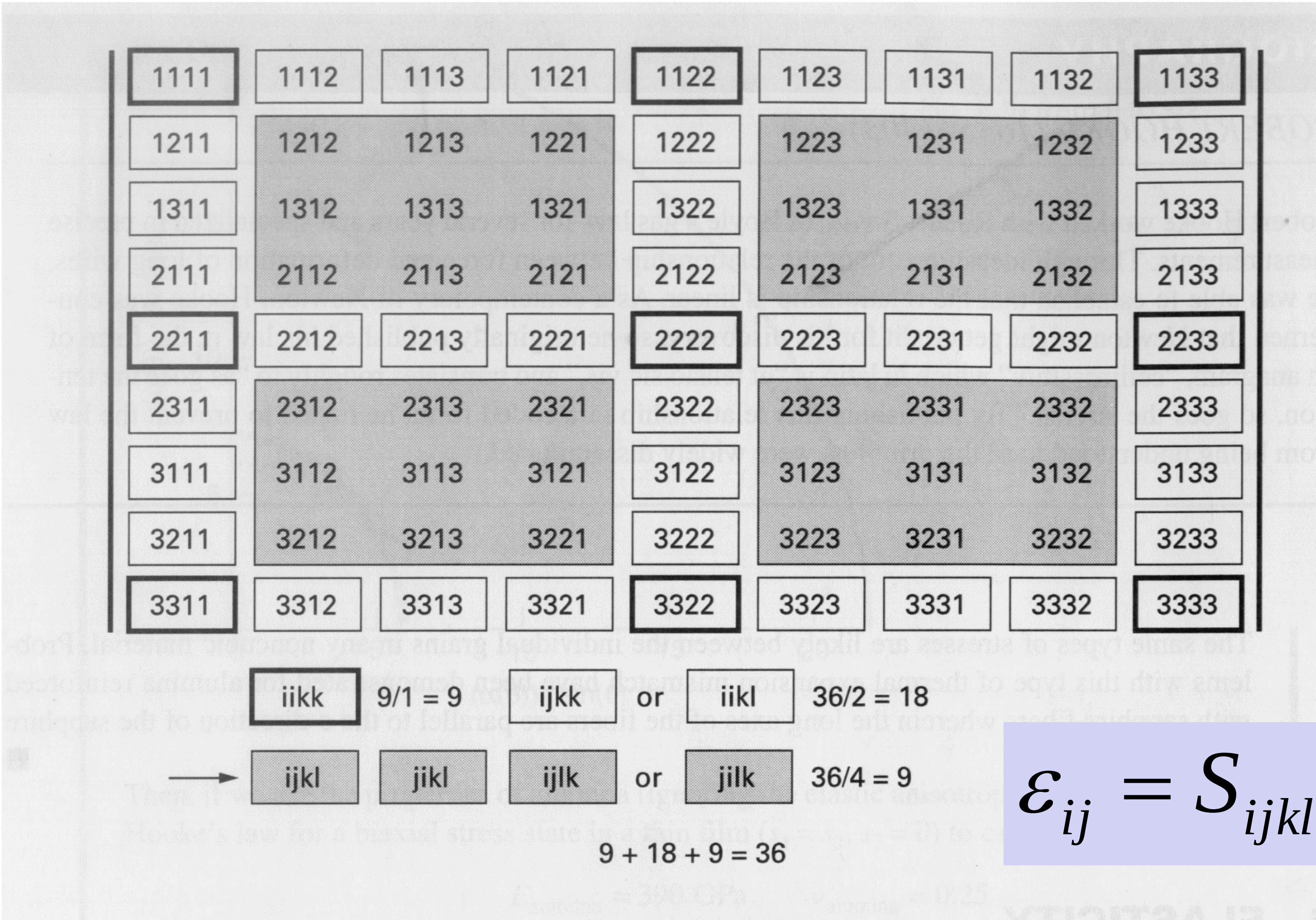
$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$ C: stiffness

$$\varepsilon_{11} = S_{1111}\sigma_{11} + S_{1112}\sigma_{12} + S_{1113}\sigma_{13}$$
$$+ S_{1121}\sigma_{21} + S_{1122}\sigma_{22} + S_{1123}\sigma_{23}$$
$$+ S_{1131}\sigma_{31} + S_{1132}\sigma_{32} + S_{1133}\sigma_{33}$$

Four-rank tensor

$\Rightarrow$ $\varepsilon_{ij} = S_{ijkl}\sigma_{kl}$ or $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

Four-rank tensor S_{ijkl}



$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl}$$

Symmetry of Four-rank tensor S

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl}$$

$$\varepsilon_{11} = S_{1111}\sigma_{11} + S_{1112}\sigma_{12} + S_{1113}\sigma_{13}$$
$$+ S_{1121}\sigma_{21} + S_{1122}\sigma_{22} + S_{1123}\sigma_{23}$$
$$+ S_{1131}\sigma_{31} + S_{1132}\sigma_{32} + S_{1133}\sigma_{33}$$

$$S_{ijkl} = S_{ijlk}$$

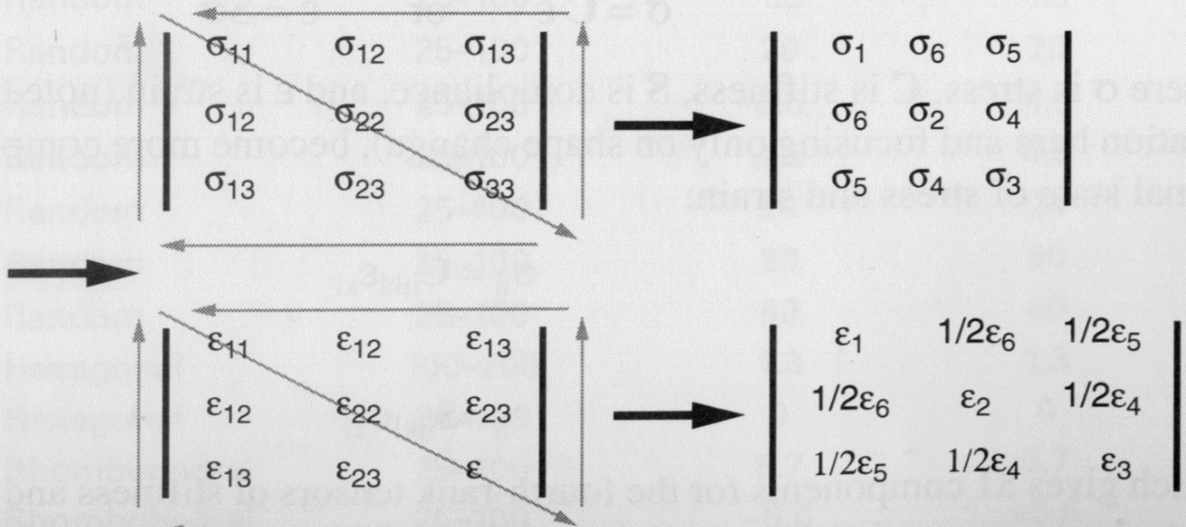
$$S_{ijkl} = S_{jikl}$$

Symmetry of Four-rank tensor S

to the matrix form that is often used to describe it,

$$\begin{vmatrix} 11 & 12 & 13 & 14 & 15 & 16 \\ 21 & 22 & 23 & 24 & 25 & 26 \\ 31 & 32 & 33 & 34 & 35 & 36 \\ 41 & 42 & 43 & 44 & 45 & 46 \\ 51 & 52 & 53 & 54 & 55 & 56 \\ 61 & 62 & 63 & 64 & 65 & 66 \end{vmatrix}$$

using the following rules for stress and strain.



Hooke's Law for Anisotropic Materials

$$\varepsilon_i = S_{ij}\sigma_j$$

S: compliance

$$\sigma_i = C_{ij}\varepsilon_j$$

C: stiffness

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix}$$

Compliance Matrix [S] for General Material

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix}$$

Stiffness Matrix [C] for General Material

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix}$$

Stiffness matrix [C] has 36 constants

Stiffness Matrix C for Monoclinic Materials

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix}$$

Stiffness matrix C for Orthorhombic (222)

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}$$

Stiffness matrix C for Tetragonal

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{11} & C_{13} & 0 & 0 & -C_{16} \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ C_{16} & -C_{16} & 0 & 0 & 0 & C_{66} \end{bmatrix}$$

Stiffness matrix C for Hexagonal

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}$$

$$C_{66} = \frac{C_{11} - C_{12}}{2}$$

Compliance matrix S for Hexagonal

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{13} & 0 & 0 & 0 \\ S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix}$$

$$S_{66} = S_{11} - S_{12}$$

Stiffness matrix C for Cubic

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix}$$

$$C_{44} = C_{55} = C_{66} = \frac{C_{11} - C_{12}}{2}$$

Compliance matrix S for Cubic

$$\begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{44} \end{bmatrix}$$

$$S_{44} = S_{55} = S_{66} = 2\left(S_{11} - S_{12}\right)$$

Form of the (s_{ij}) and (c_{ij}) matrices

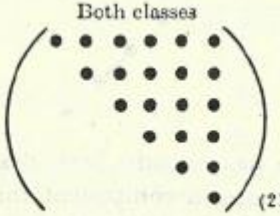
KEY TO NOTATION

- · zero component
- ● non-zero component
- ●—● equal components
- ●—○ components numerically equal, but opposite in sign
- For s ◎ twice the numerical equal of the heavy dot component to which it is joined
- For c ◎ the numerical equal of the heavy dot component to which it is joined
- For s X $2(s_{11}-s_{12})$
- For c X $\frac{1}{2}(c_{11}-c_{12})$

All the matrices are symmetrical about the leading diagonal.

Triclinic

Both classes

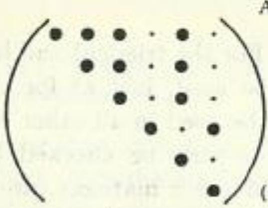


(21)

Monoclinic

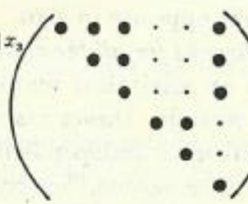
All classes

Diad $\parallel x_2$ (standard orientation)



(13)

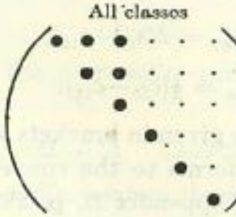
Diad $\parallel x_3$



(13)

Orthorhombic

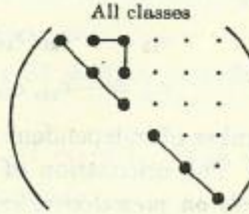
All classes



(9)

Cubic

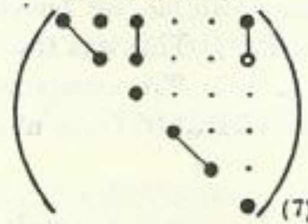
All classes



(3)

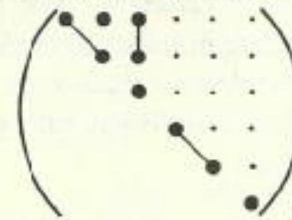
Tetragonal

Classes $4, \bar{4}, 4/m$



(7)

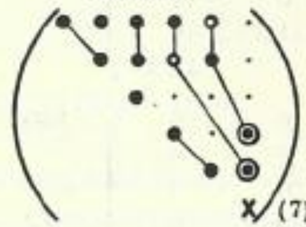
Classes $4mm, \bar{4}2m,$† $422, 4/mmm$



(6)

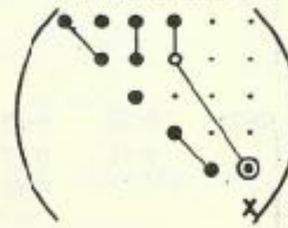
Trigonal

Classes $3, \bar{3}$



(7)

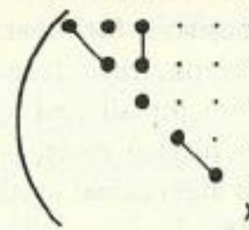
Classes $32, \bar{3}m, 3m$



(6)

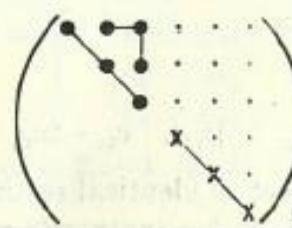
Hexagonal

All classes



(5)

Isotropic



(2)

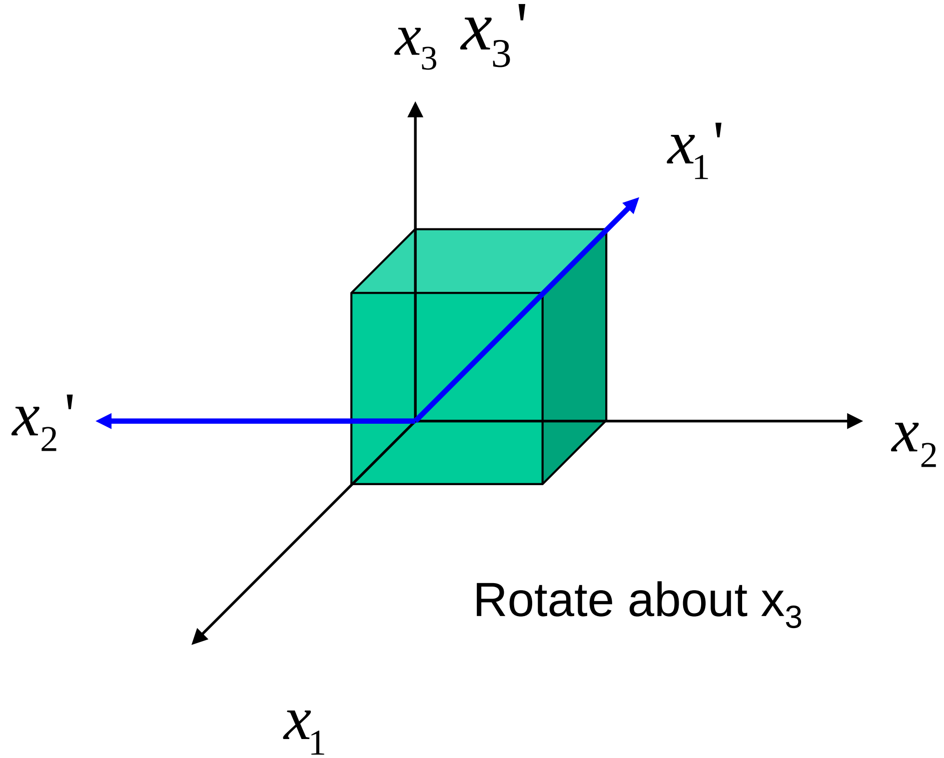
Crystal system	Laue class	Point groups	C_{ij}'s
Triclinic	$\bar{1}$	$1, \bar{1}$	21
Monoclinic	$2/m$	$2, m, 2/m$	13
Orthorhombic	mmm	$222, 2mm, mmm$	9
Tetragonal (II)	$4/m$	$4, \bar{4}, 4/m$	7
Tetragonal (I)	$4/mmm$	$4mm, 422, \bar{4}2m, 4/mmm$	6
Rhombohedral (II)	$\bar{3}$	$3, \bar{3}$	7
Rhombohedral (I)	$\bar{3}m$	$32, 3m, \bar{3}m$	6
Hexagonal (II)	$6/m$	$6, \bar{6}, 6/m$	5
Hexagonal (I)	$6/mmm$	$6mm, 622, \bar{6}2m, 6/mmm$	5
Cubic (II)	$m\bar{3}$	$23, m\bar{3}$	3
Cubic (I)	$m\bar{3}m$	$432, \bar{4}3m, m\bar{3}m$	3

Table I. Laue groups and number of independent second-order elastic constants C_{ij}. We follow the naming convention of Wallace[5] (I/II) to distinguish Laue classes within the same crystal system.

Demonstration of twofold symmetry on stiffness/ compliance

Two-fold rotation	in simplified notation	example
$x_1 \to -x_1'$	$1 \to -1$	$\sigma_{11} \to \sigma_{11}$
$x_2 \to -x_2'$	$2 \to -2$	$\sigma_{13} \to -\sigma_{13}$
$x_3 \to x_3'$	$3 \to 3$	

$11 \to 11$	$1 \to 1$
$22 \to 22$	$2 \to 2$
$33 \to 33$	$3 \to 3$
$23 \to -23$	$4 \to -4$
$13 \to -13$	$5 \to -5$
$12 \to 12$	$6 \to 6$



Rotate about x_3

Twofold symmetry on stiffness (Monoclinic system)

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{pmatrix}$$

$$1 \rightarrow 1 \quad 4 \rightarrow -4$$
$$2 \rightarrow 2 \quad 5 \rightarrow -5$$
$$3 \rightarrow 3 \quad 6 \rightarrow 6$$

$$\Rightarrow$$

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & -C_{14} & -C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & -C_{24} & -C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & -C_{34} & -C_{35} & C_{36} \\ -C_{41} & -C_{42} & -C_{43} & C_{44} & C_{45} & -C_{46} \\ -C_{51} & -C_{52} & -C_{53} & C_{54} & C_{55} & -C_{56} \\ C_{61} & C_{62} & C_{63} & -C_{64} & -C_{65} & C_{66} \end{pmatrix}$$

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{pmatrix}$$

$$C_{14} = -C_{14} = 0$$
$$C_{15} = -C_{15} = 0$$
$$C_{24} = -C_{24} = 0$$
$$C_{25} = -C_{25} = 0$$

Twofold symmetry on stiffness (Orthorhombic system)

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{21} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{31} & C_{32} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{54} & C_{55} & 0 \\ C_{61} & C_{62} & C_{63} & 0 & 0 & C_{66} \end{pmatrix}$$

R x_3	R x_2	R x_1
$1 \to -1$	$1 \to -1$	$1 \to 1$
$2 \to -2$	$2 \to 2$	$2 \to -2$
$3 \to 3$	$3 \to -3$	$3 \to -3$

$1 \to 1$	$1 \to 1$	$1 \to 1$
$2 \to 2$	$2 \to 2$	$2 \to 2$
$3 \to 3$	$3 \to 3$	$3 \to 3$
$4 \to -4$	$4 \to -4$	$4 \to 4$
$5 \to -5$	$5 \to 5$	$5 \to -5$
$6 \to 6$	$6 \to -6$	$6 \to -6$

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{pmatrix}$$

Demonstration of fourfold symmetry on stiffness (Tetragonal system $\bar{4}$)

$$
\begin{pmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\
C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\
C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}
\end{pmatrix}
$$

$$
\begin{pmatrix}
C_{22} & C_{21} & C_{23} & C_{25} & -C_{24} & -C_{26} \\
C_{12} & C_{11} & C_{13} & C_{25} & -C_{14} & -C_{16} \\
C_{32} & C_{31} & C_{33} & C_{35} & -C_{34} & -C_{36} \\
C_{52} & C_{51} & C_{53} & C_{55} & -C_{54} & -C_{56} \\
-C_{42} & -C_{41} & -C_{43} & -C_{45} & C_{44} & C_{46} \\
-C_{62} & -C_{61} & -C_{63} & -C_{65} & C_{64} & C_{66}
\end{pmatrix}
$$

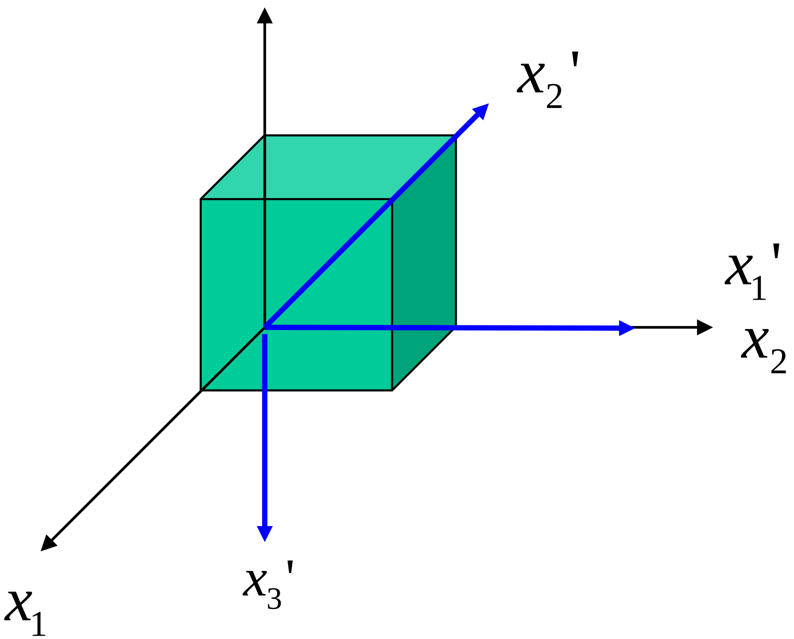
R x_3

$1 \rightarrow 2$

$2 \rightarrow -1$

$3 \rightarrow -3$

$11 \rightarrow 22$	$1 \rightarrow 2$
$22 \rightarrow 11$	$2 \rightarrow 1$
$33 \rightarrow 33$	$3 \rightarrow 3$
$23 \rightarrow 13$	$4 \rightarrow 5$
$13 \rightarrow -23$	$5 \rightarrow -4$
$12 \rightarrow -21$	$6 \rightarrow -6$



Fourfold symmetry on stiffness (Tetragonal system)

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{pmatrix}$$

$$\begin{pmatrix} C_{22} & C_{21} & C_{23} & C_{25} & -C_{24} & -C_{26} \\ C_{12} & C_{11} & C_{13} & C_{25} & -C_{14} & -C_{16} \\ C_{32} & C_{31} & C_{33} & C_{35} & -C_{34} & -C_{36} \\ C_{52} & C_{51} & C_{53} & C_{55} & -C_{54} & -C_{56} \\ -C_{42} & -C_{41} & -C_{43} & -C_{45} & C_{44} & C_{46} \\ -C_{62} & -C_{61} & -C_{63} & -C_{65} & C_{64} & C_{66} \end{pmatrix}$$

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{11} & C_{13} & 0 & 0 & -C_{16} \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ C_{16} & -C_{16} & 0 & 0 & 0 & C_{66} \end{pmatrix}$$

$$C_{15} = -C_{24} = 0$$

$$C_{26} = -C_{16}$$

$$C_{25} = -C_{24} = 0$$

$$C_{35} = -C_{34} = 0$$

$$C_{45} = -C_{54} and C_{45} = C_{54}$$

$$C_{54} = -C_{45} and C_{54} = C_{45}$$

Relations among Elastic Constants for Isotropic Materials

Table 2.2 Relations among the Elastic Constants for Isotropic Materials

Elastic Constants	In Terms of: E, ν	E, G	K, ν	K, G	λ, μ
E	$=E$	$=E$	$=3(1-2\nu)K$	$=\dfrac{9K}{1+3K/G}$	$=\dfrac{\mu(3+2\mu/\lambda)}{1+\mu/\lambda}$
ν	$=\nu$	$=-1+\dfrac{E}{2G}$	$=\nu$	$=\dfrac{1-2G/3K}{2+2G/3K}$	$=\dfrac{1}{2(1+\mu/\lambda)}$
G	$=\dfrac{E}{2(1+\nu)}$	$=G$	$=\dfrac{3(1-2\nu)K}{2(1+\nu)}$	$=G$	$=\mu$
K	$=\dfrac{E}{3(1-2\nu)}$	$=\dfrac{E}{9-3E/G}$	$=K$	$=K$	$=\lambda+\dfrac{2\mu}{3}$
λ	$=\dfrac{E\nu}{(1+\nu)(1-2\nu)}$	$=\dfrac{E(1-2G/E)}{3-E/G}$	$=\dfrac{3K\nu}{1+\nu}$	$=K-\dfrac{2G}{3}$	$=\lambda$
μ	$=\dfrac{E}{2(1+\nu)}$	$=G$	$=\dfrac{3(1-2\nu)K}{2(1+\nu)}$	$=G$	$=\mu$

Elastic Compliance and Stiffness Matrixes

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(S_{11} - S_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(S_{11} - S_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(S_{11} - S_{12}) \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix}$$

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & (C_{11} - C_{12})/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (C_{11} - C_{12})/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (C_{11} - C_{12})/2 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix}$$

Elastic Compliance and Stiffness Matrixes

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(S_{11}-S_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix}$$

$$\varepsilon_1 = S_{11}\sigma_1 + S_{12}\sigma_2 + S_{12}\sigma_3 = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right]$$

$$\varepsilon_{22} = S_{12}\sigma_1 + S_{11}\sigma_2 + S_{12}\sigma_3 = \frac{1}{E}\left[\sigma_2 - \nu(\sigma_1 + \sigma_3)\right]$$

$$\varepsilon_{33} = S_{12}\sigma_1 + S_{12}\sigma_2 + S_{11}\sigma_3 = \frac{1}{E}\left[\sigma_3 - \nu(\sigma_1 + \sigma_2)\right]$$

$$\varepsilon_4 = 2(S_{11} - S_{12})\sigma_4 = \frac{1}{G}\sigma_4$$

$$\varepsilon_5 = 2(S_{11} - S_{12})\sigma_5 = \frac{1}{G}\sigma_5$$

$$\varepsilon_6 = 2(S_{11} - S_{12})\sigma_6 = \frac{1}{G}\sigma_6$$

Elastic Compliance and Stiffness Matrixes

$$S_{11}\left[\sigma_1 + \frac{S_{12}}{S_{11}}(\sigma_2 + \sigma_3)\right] = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right]$$

Young's modulus $E = \frac{1}{S_{11}}$

Poisson's ratio $\nu = -\frac{S_{12}}{S_{11}}$

$$2(S_{11} - S_{12})\sigma_4 = \frac{\sigma_4}{G}$$

Shear modulus

$$G = \frac{1}{2(S_{11} - S_{12})}$$

Relationships Among Elastic Constants

Young's modulus

$$E = \frac{1}{S_{11}}$$

Shear modulus

$$G = \frac{1}{2\left(S_{11} - S_{12}\right)}$$

Bulk modulus

$$B = \frac{1}{K} = \frac{\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}}{-1/3\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)}$$

Poisson's ratio

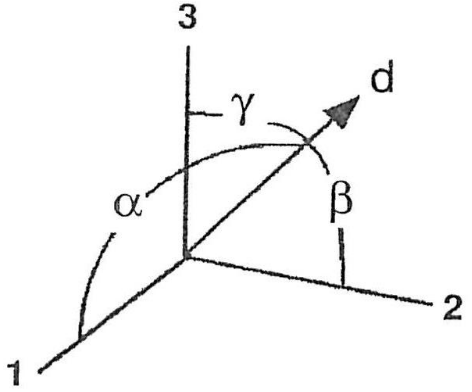
$$\nu = -\frac{S_{12}}{S_{11}}$$

Lame′ constants

$$\mu = C_{44} = \frac{1}{2}\left(C_{11} - C_{12}\right) \qquad \lambda = C_{12}$$

Orientation dependence in cubic crystals

$$\sigma_1 = \sigma^2 \sigma_d \quad \sigma_{23} = \beta\gamma\sigma_d$$
$$\sigma_2 = \beta^2 \sigma_d \quad \sigma_{31} = \gamma\alpha\sigma_d$$
$$\sigma_3 = \gamma^2 \sigma_d \quad \sigma_{12} = \alpha\beta\sigma_d$$



$$\varepsilon_1/\sigma_d = s_{11}\alpha^2 + s_{12}\beta^2 + s_{12}\gamma^2, \quad \gamma_{23}/\sigma_d = s_{44}\beta\gamma$$
$$\varepsilon_2/\sigma_d = s_{12}\alpha^2 + s_{11}\beta^2 + s_{12}\gamma^2, \quad \gamma_{31}/\sigma_d = s_{44}\gamma\alpha$$
$$\varepsilon_3/\sigma_d = s_{12}\alpha^2 + s_{12}\beta^2 + s_{11}\gamma^2, \quad \gamma_{12}/\sigma_d = s_{44}\alpha\beta$$

$$\varepsilon_d/\sigma_d = \alpha^2 e_1 + \beta^2 e_2 + \gamma^2 e_3 + \beta\gamma\cdot\gamma_{23} + \gamma\alpha\cdot\gamma_{31} + \alpha\beta\cdot\gamma_{12}$$
$$e_d/\sigma_d = s_{11}\left(\alpha^4 + \beta^4 + \gamma^4\right) + 2s_{12}\left(\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2\right) + s_{44}\left(\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2\right)$$
$$= s_{11}\left(\alpha^4 + \beta^4 + \gamma^4\right) + \left(2s_{12} + s_{44}\right)\left(\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2\right)$$

Use $\alpha^2 + \beta^2 + \gamma^2 = 1$

$$1/E_d = \varepsilon_d/\sigma_d = s_{11} + \left(-2s_{11} + 2s_{12} + s_{44}\right)\left(\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2\right)$$

$$\frac{1}{E_{ijk}} = S_{11} - 2\left(S_{11} - S_{11} - \frac{1}{2}S_{11}\right)\left(l_{i1}^2 l_{j2}^2 + l_{j2}^2 l_{k3}^2 + l_{k3}^2 l_{i1}^2\right)$$

Orientation dependence in cubic crystals-conti.

$$1/E_d = e_d/\sigma_d = s_{11} + (-2s_{11} + 2s_{12} + s_{44})(\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2)$$

Express d as [*hkl*]

$$\alpha = h/\sqrt{h^2+k^2+l^2},\ \beta = k/\sqrt{h^2+k^2+l^2},\ \gamma = l/\sqrt{h^2+k^2+l^2}$$

$$1/E_d = s_{11} + (2s_{12} - 2s_{11} + s_{44})(k^2l^2 + l^2h^2 + h^2k^2)/(h^2+k^2+l^2)^2$$

Because of isotropic ➔ $s_{44} = 2(s_{11} - s_{12})$

$$E_{[111]}/E_{[100]} = 3s_{11}/(s_{11} + 2s_{12} + s_{44})$$

$$1/E_d = 1/E_{100} + f\,[1/E_{111} - 1/E_{100}]$$

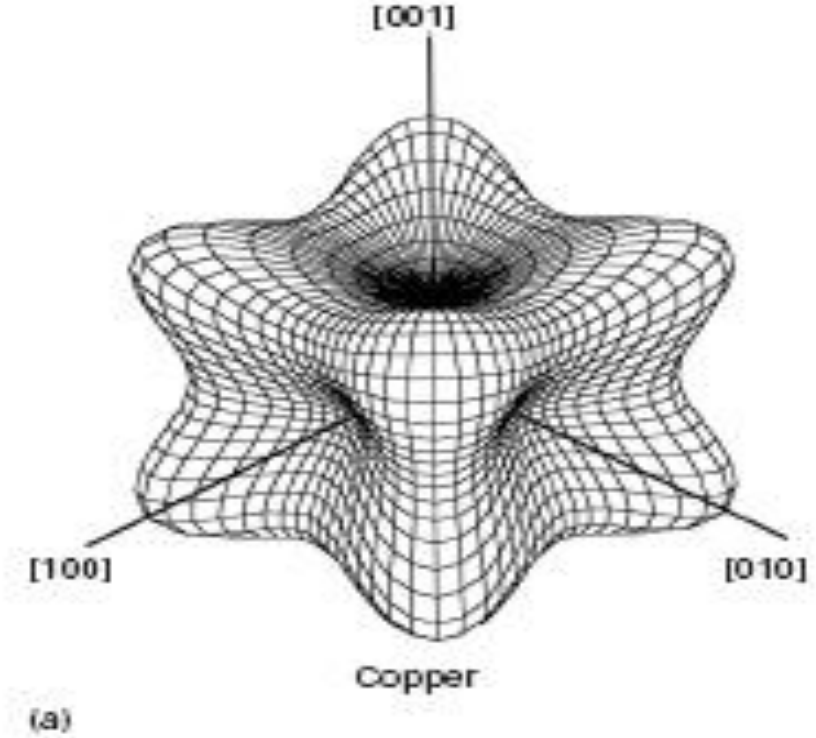
Elastic modus in terms of orientation for cubic

$$\frac{1}{E_{ijk}} = S_{11} - 2\left(S_{11} - S_{12} - \tfrac{1}{2}S_{44}\right) \times \left(l_{i1}^2 l_{j2}^2 + l_{j2}^2 l_{k3}^2 + l_{k3}^2 l_{i1}^2\right)$$

l_{i1}, l_{j2}, l_{k3} the direction consines of the direction [I,j,k]

	l_{i1}	l_{j2}	l_{k3}	$\left(l_{i1}^2 l_{j2}^2 + l_{j2}^2 l_{k3}^2 + l_{k3}^2 l_{i1}^2\right)$
[100]	1	0	0	0
[110]	$1/\sqrt{2}$	$1/\sqrt{2}$	0	$1/4$
[111]	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/3$

Young's Modulus of single crystalline Cu (Cubic)



$$S_{11} = 1.525 \times 10^{-2} GPa^{-1}$$

$$S_{44} = 1.323 \times 10^{-2} GPa^{-1}$$

$$S_{12} = -0.639 \times 10^{-2} GPa^{-1}$$

$$E_{100} = 66GPa$$

$$E_{110} = 130GPa$$

$$E_{111} = 191GPa$$

Elastic modus of polycrystals

$$\overline{E}(Voigt) = \frac{(C_{11} + 2C_{12})(C_{11} - C_{12} + 3C_{44})}{(2C_{11} + 3C_{12} + C_{44})}$$

$$\overline{E}(\mathrm{Re}\,uss) = \frac{5}{3S_{11} + 2S_{12} + S_{44}}$$

Anisotropy factor $A = \frac{C_{44}}{(C_{11} - C_{12})/2}$

$\overline{E}(Voigt) = 144.6GPa$ $\overline{E}(\mathrm{Re}\,uss) = 108GPa$

$$A(Cu) = 3.2$$