Engineering Mechanics of Materials

Chapter 02

Axially Loaded Members

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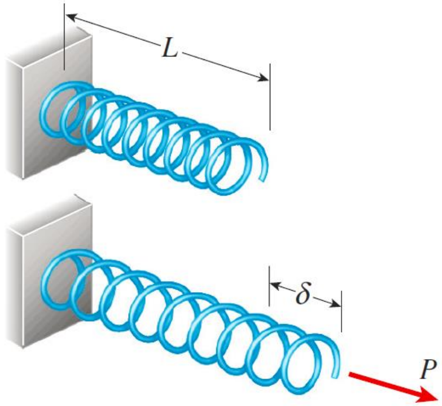
Learning Objectives

- Study changes in length of axially loaded members based on a force-displacement relation.
- Find support reactions in statically indeterminate bars acted on by concentrated and distributed axial forces.
- Find changes in lengths of bars due to temperature, misfit, and pre-strain effects.
- Find both normal and shear stresses on inclined section at points of interest on axially loaded bars.
- Study the effects of holes through axially loaded bars that cause localized stress concentrations.
- Study selected advanced topics such as strain energy, impact, fatigue, and nonlinear behavior.

Coiled Springs

- Tension– load acts <u>away</u>; spring <u>elongates</u>
- Compression– load acts <u>towards</u>; spring <u>shortens</u>



- Linearly Elastic Equations

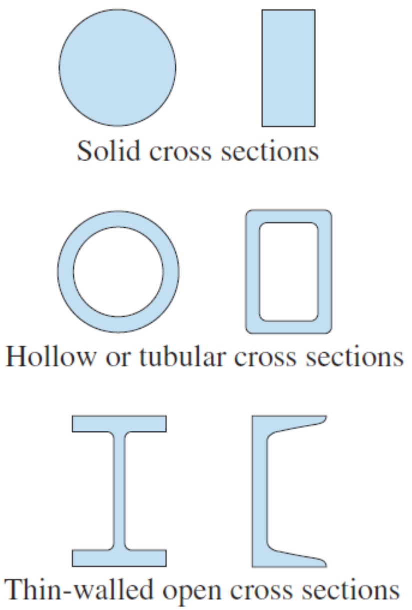
Load: $P = k\delta$ Elongation: $\delta = fP$

Stiffness: $k = \frac{P}{\delta} = \frac{1}{f}$ Flexibility: $f = \frac{\delta}{P} = \frac{1}{k}$

Prismatic Bars

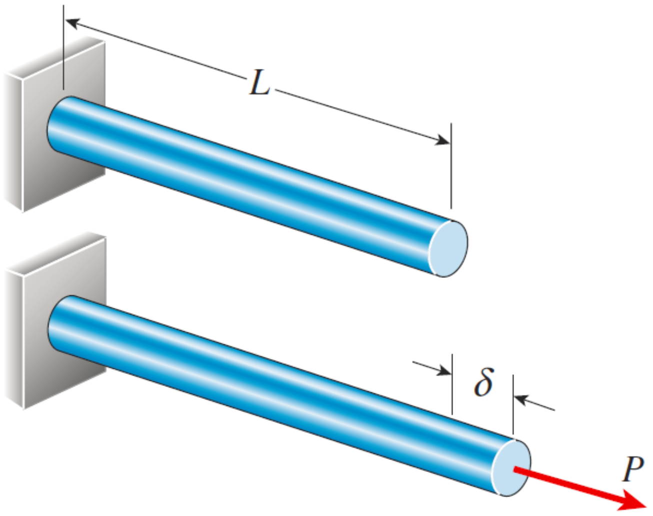
- Requirement #1: Straight longitudinal axis
- Requirement #2: Constant cross section
  - Geometry can vary (see image)
- Requirement #3: Linearly elastic
- Force-Displacement Relation

$$\boxed{\sigma = \frac{P}{A}}, \quad \boxed{\varepsilon = \frac{\delta}{L}} \quad \text{Substitute} \rightarrow \boxed{\sigma = E\varepsilon} \rightarrow \text{Yields} \rightarrow \delta = \frac{PL}{EA}$$



Prismatic Bars (cont.)

- Deformation Sign Convention
  - Elongation: Positive (+)
  - Shortening: Negative (-)



- Stiffness: $k = \dfrac{P}{\delta} \longrightarrow k = \dfrac{EA}{L}$

- Flexibility: $f = \dfrac{\delta}{P} \longrightarrow f = \dfrac{L}{EA}$

Cables

- Tension only (cannot resist compression)
- δ cable > δ prismatic-bar for the same load, material, and cross-section.
- Modulus of elasticity for a cable < modulus of elasticity of the material.
- In analysis, use the cable's effective modulus and not for the material.

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Bars with Intermediate Axial Loads

- Step 1: Identify segments (ex: AB, BC, and CD as 1, 2, and 3).

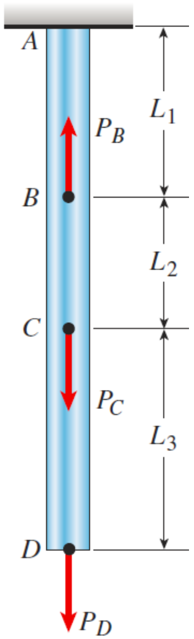
- Step 2: Determine the internal axial forces N_1, N_2, and N_3.
  - Make a sectional cut at each segment and evaluate FBD.

- Step 3: Determine elongation for each segment.

Ex: $\delta_1 = \dfrac{N_1 L_1}{EA} \qquad \delta_2 = \dfrac{N_2 L_2}{EA} \qquad \delta_3 = \dfrac{N_3 L_3}{EA}$

- Step 4: Add elongation vales for overall elongation.

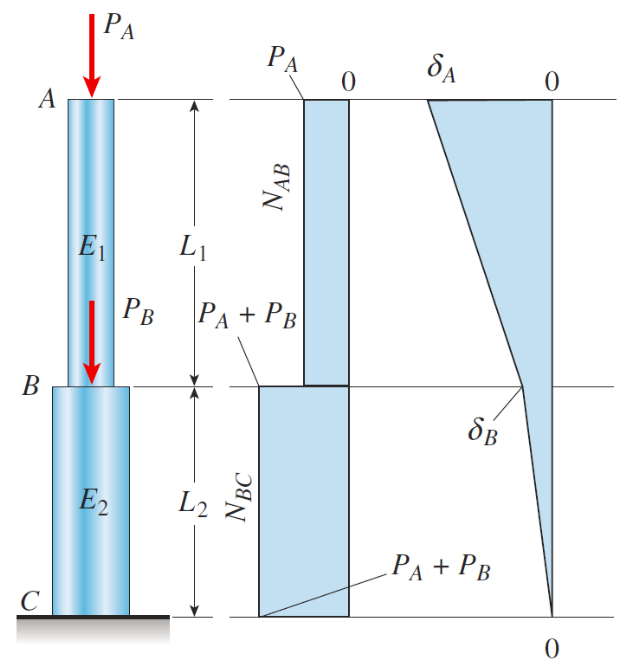
Ex: $\delta_{bar} = \sum_{i=1}^{3} \delta_i = \delta_1 + \delta_2 + \delta_3$



Bars Consisting of Prismastic Segments

- Analysis follows the same procedure.
- However, EA are no longer constants.
- General Elongation Equation

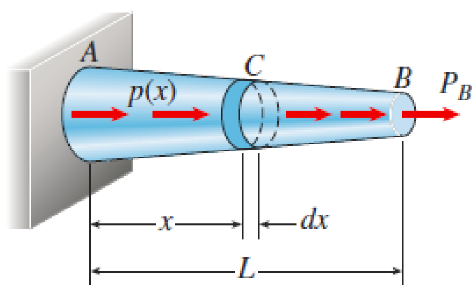
$$\delta = \sum_{i=1}^{n} \frac{N_i L_i}{E_i A_i}$$

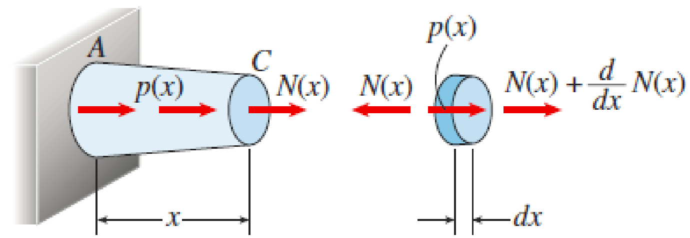


Bars with Continuously Varying Loads or Dimensions

- Condition #1: Continuously varying cross-sectional area A.
- Condition #2: Continuously varying axial force N.
- Goal: Determine the elongation of a differential element of the bar and integrate over the length of the bar.
- Limitations: Must be linearly elastic with any taper angle being small.

$$\delta = \int_0^L d\delta = \int_0^L \frac{N(x)dx}{EA(x)}$$





Changes in Lengths Under Nonuniform Conditions

Graphical display of the internal axial force over the length of the bar

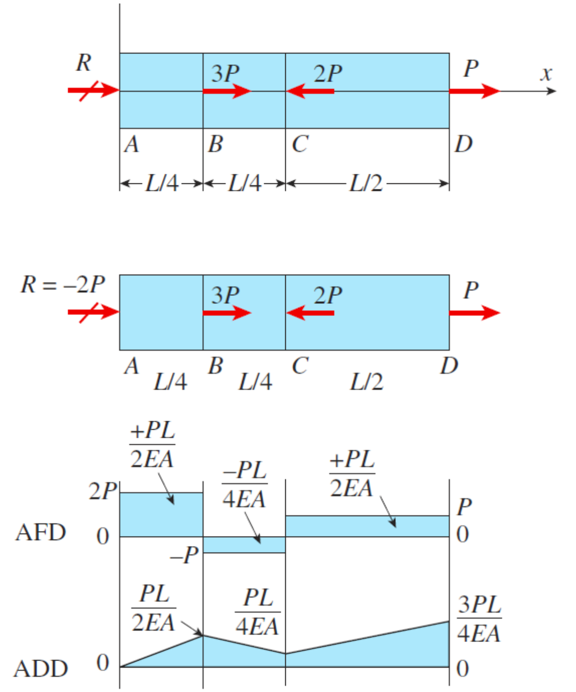
Graphical display of the Axial displacement over the length of the bar

Schematic

Overall FBD

Axial Force Diagram

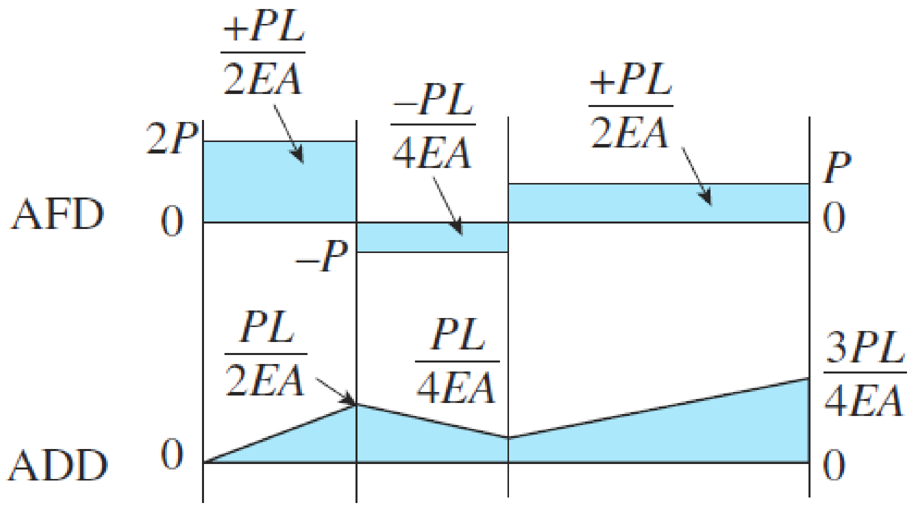
Axial Displacement Diagram



AFD to ADD Plotting Guidelines

- The slope at any point on the ADD is equal to the originate on the AFD at that same point divided by the bar's axial rigidity EA at the same location.

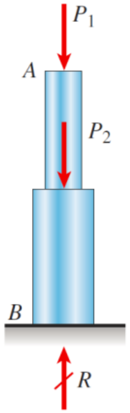
- The change in axial displacement between any two points along a bar is equal to the area under the axial force diagram between those same two points divided by the bar's axial rigidity EA over that same interval.



Statically Indeterminate Structures

Statically Determinate

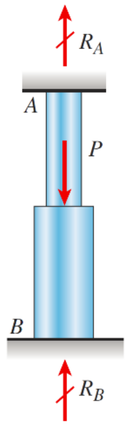
- Reactions and internal forces can be found through FBD and equilibrium equations.



$$\sum F_{vertical} = 0 \qquad R_A - P_1 - P_2 = 0$$

Statically Indeterminate

- More unknowns than the number of equations provided through FBD and equilibrium analysis alone. As such, more equations are needed.



Equation of Compatibility

$$\delta_{AB} = 0 \quad \text{or} \quad \delta_{AB} = gap$$

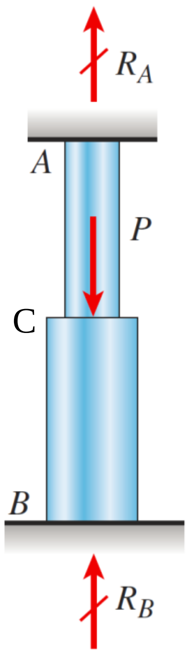
Statically Indeterminate Structures

Boundary condition: $\delta_{AB} = 0$

Solving for the displacements: $\delta_{AC} = \dfrac{R_A L_{AC}}{EA}$ $\qquad \delta_{CB} = -\dfrac{R_B L_{CB}}{EA}$

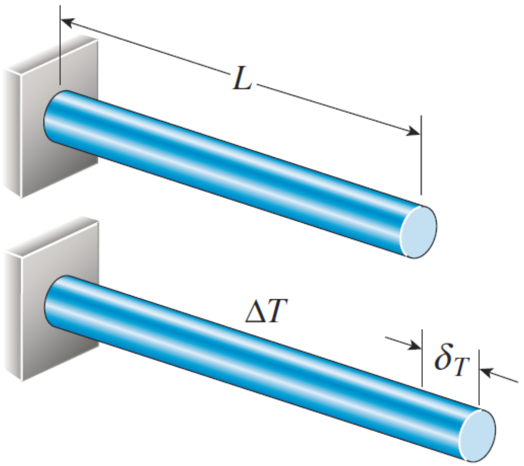
Equation of compatibility: $\delta_{AB} = \delta_{AC} + \delta_{CB} = \dfrac{R_A L_{AC}}{EA_{AC}} - \dfrac{R_B L_{CB}}{EA_{CB}} = 0$

Solving for the reactions: $R_A = \dfrac{PL_{AC}}{L_{AB}}$ $\qquad R_B = \dfrac{PL_{CB}}{L_{AB}}$



Thermal Effects

- Thermal Strain: $\varepsilon_T = \alpha(\Delta T)$
- Coefficient of Thermal Expansion: α
  - Units: Reciprocal of the temperature change
- Sign Convention:
  - Expansion: Positive (+)
  - Contraction: Negative (-)



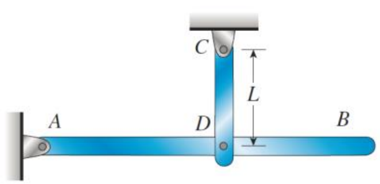
- Thermal Stress: $\sigma = E\varepsilon_T = E\alpha(\Delta T)$
- Temperature-Displacement Relation: $\delta_T = \varepsilon_T L = \alpha(\Delta T)L$

Misfits and Pre-Strains

- Misfit: Members in structure fit incorrectly due to their improper lengths.
- Pre-Strain: Strains produced in the member due to the misfit.
- Pre-Stressed: The stresses which accompanies the strains in misfits.

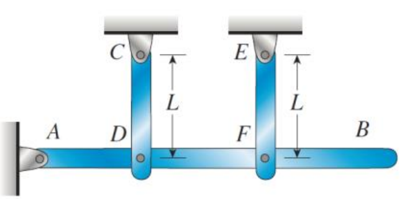
Statically Determinate

Misfits will not produce strains or stresses, though a small angle will be produced with the horizontal.



Statically Indeterminate

Pre-strains will exist in all the members and the structure will be pre-stressed even with no external loads.

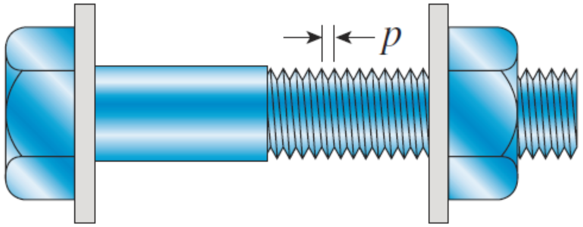


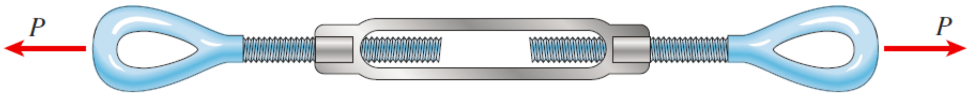
Bolts and Turnbuckles

- Bolts and turnbuckles allows one to change the length of a member
- Bolt Elongation:

$$\delta = np$$

  - n = number of revolutions
  - p = pitch of the threads

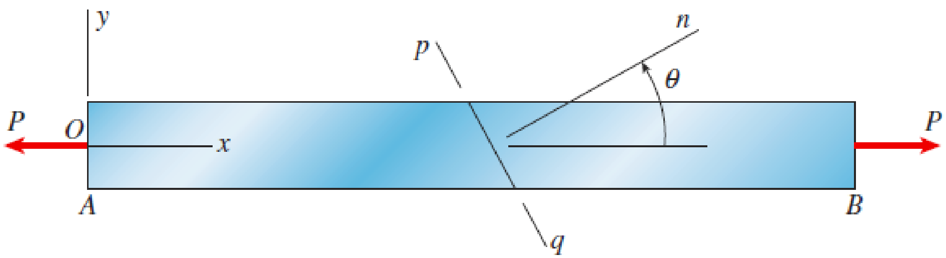




- Turnbuckle Elongation:

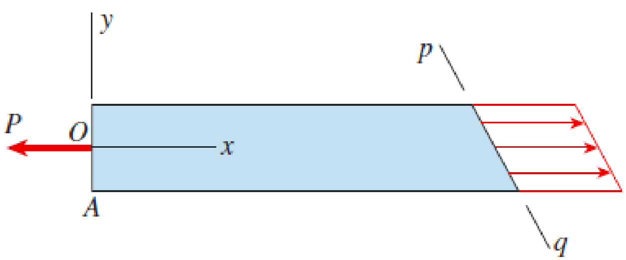
$$\delta = 2np$$

Stresses on Inclined Sections

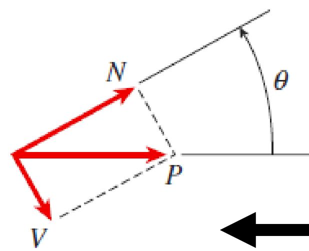


(a)

Sectional cut at inclined plane PQ for an axially loaded bar



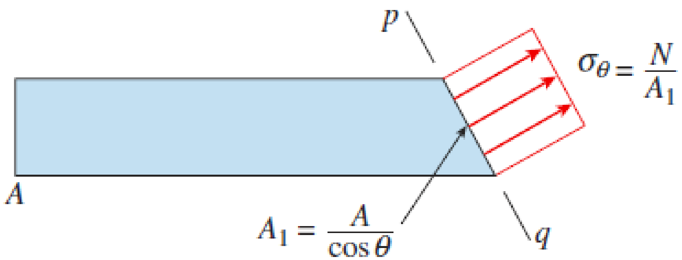
(b)



$$\boxed{N = P\cos\theta}$$

$$\boxed{V = P\sin\theta}$$

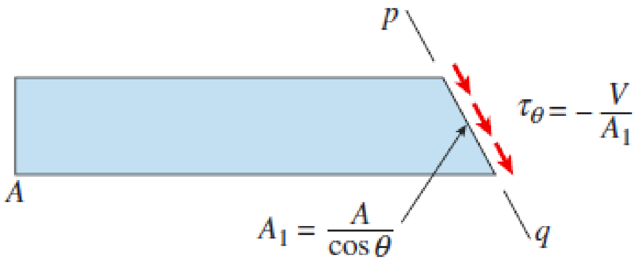
Stresses on Inclined Sections (cont.)



(c)

$$\sigma_\theta = \frac{N}{A_{incline}} = \frac{P}{A_{cross}} \cos^2\theta$$

$$A_{incline} = \frac{A_{cross}}{\cos\theta}$$



(d)

$$\tau_\theta = \frac{V}{A_{incline}} = -\frac{P}{A_{cross}} \sin\theta\cos\theta$$

Stresses on Inclined Sections (cont.)

Applying these three relationships:

$$\sigma_x = \frac{P}{A_{cross}} \qquad \sin\theta\cos\theta = \frac{1}{2}(\sin 2\theta) \qquad \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

Normal Inclined Stress

$$\sigma_\theta = \sigma_x \cos^2\theta = \frac{\sigma_x}{2}(1+\cos 2\theta)$$

Shear Inclined Stress

$$\tau_\theta = -\sigma_x \sin\theta\cos\theta = -\frac{\sigma_x}{2}(\sin 2\theta)$$

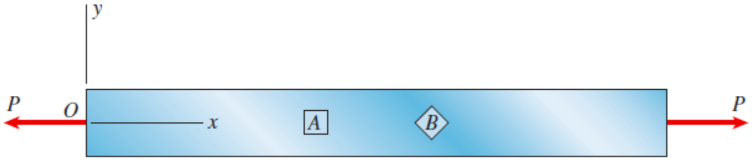
Stresses on Inclined Sections (cont.)

- Maximum Normal Stress
  - Occurs at $\theta = 0°$

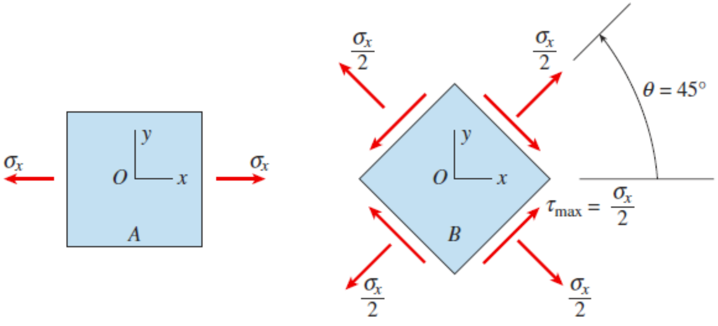
$$\sigma_{\max} = \sigma_x$$

- Maximum Shear Stress
  - Occurs at $\theta = \pm 45°$

$$\tau_{\max} = \frac{\sigma_x}{2}$$



(a)



(b) (c)

Strain Energy

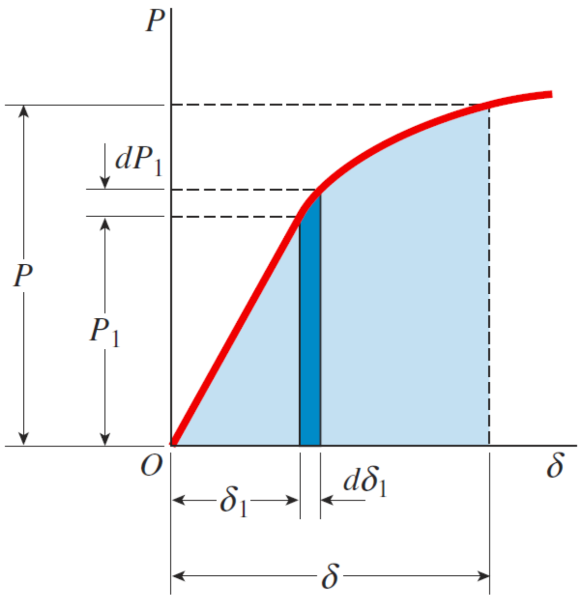
- Work– the area below the load-displacement curve

$$W = \int_0^{\delta} P d\delta$$

- Strain Energy– the energy absorbed by the bar during loading

$$U = W = \int_0^{\delta} P d\delta$$

- Units: <u>SI</u> → J, N·m <u>IP</u> → ft-lb, ft-kips, in-lb, in-kips



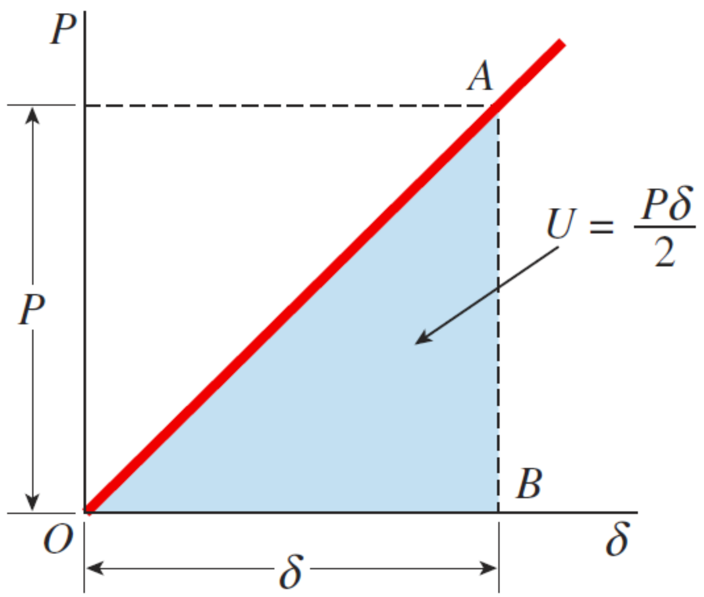
Strain Energy (cont.)

Linearly Elastic Behavior:

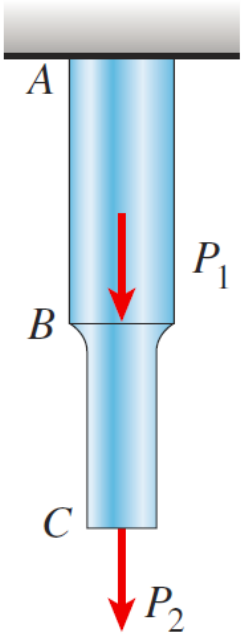
$$\boxed{U_{linear} = W_{linear} = \frac{P\delta}{2}} \rightarrow \text{sub} \rightarrow \boxed{\delta = \frac{PL}{EA}}$$

$$\boxed{U_{linear} = \frac{P^2 L}{2EA} = \frac{EA\delta^2}{2L}}$$

Linearly Elastic Spring: $U_{spring} = \dfrac{P^2}{2k} = \dfrac{k\delta^2}{2}$



Strain Energy (cont.)

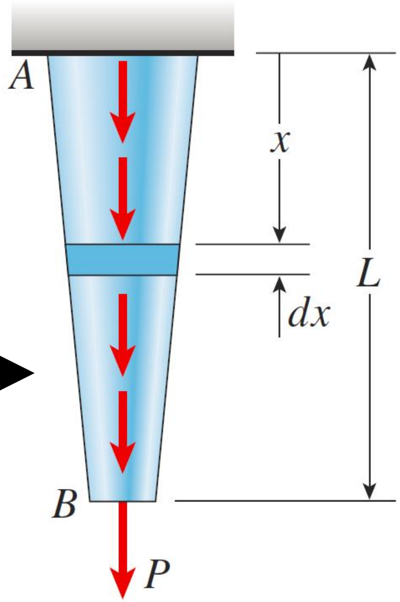


Segmented Bars

$$U_{segmented} = \sum_{i=1}^{n} U_i = \sum_{i=1}^{n} \frac{N_i^{\,2} L_i}{2E_i A_i}$$

Non-Prismatic Bars

$$U_{non-prismatic} = \int_0^L \frac{[N(x)]^2\,dx}{2EA(x)}$$

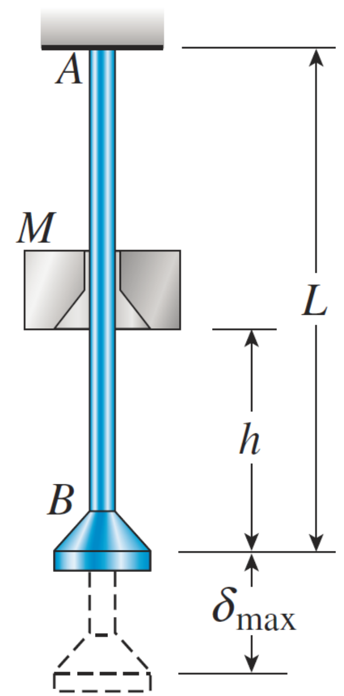


Maximum Elongation of the Bar

- From conservation of energy, one equates potential energy lost by a falling mass to the maximum strain energy acquired by the bar:

$$W(h+\delta_{\max}) = \frac{EA\delta^2_{\max}}{2L} \longrightarrow \delta_{\max} = \sqrt{\frac{Mv^2L}{EA}}$$

Where: $M = \dfrac{W}{g}$ and $v = \sqrt{2gh}$

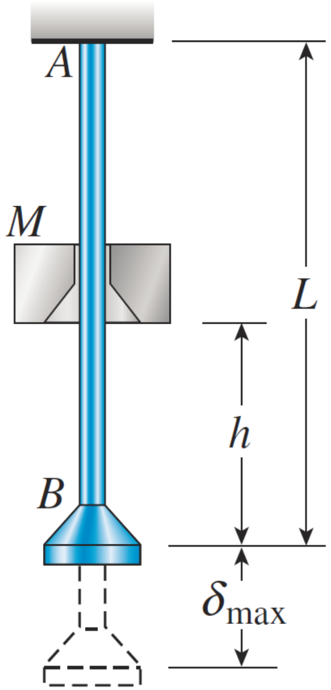


Maximum Stress in the Bar

- From the maximum elongation, the stress distribution is assumed to be uniform throughout the bar's length. Therefore, the force-displacement relation is applied.

$$\sigma_{\max} = \frac{E\delta_{\max}}{L} \quad \longrightarrow \quad \sigma_{\max} = \sqrt{\frac{Mv^2E}{AL}}$$

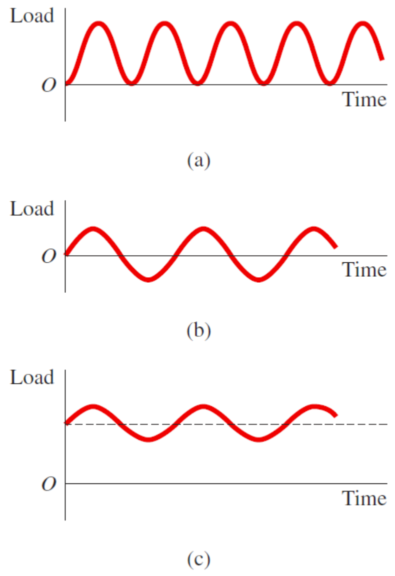
Where: $M = \dfrac{W}{g}$ and $v = \sqrt{2gh}$



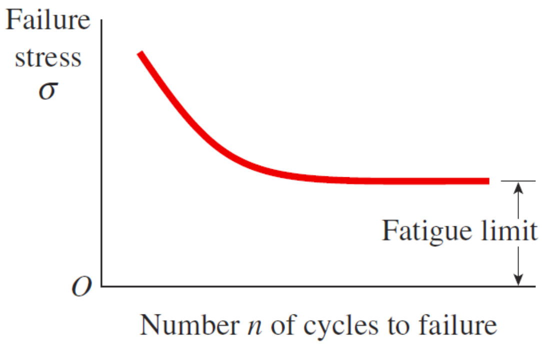
Repeated Loading and Fatigue

- Fatigue failure will result due to repeated loading.
- Cracks form at a stress concentration.
- Endurance curves (S-N diagrams) plot failure stress (S) against the number of cycles (N) to failure.
- Fatigue limit is the value at which lower stresses will not produce a fatigue failure.

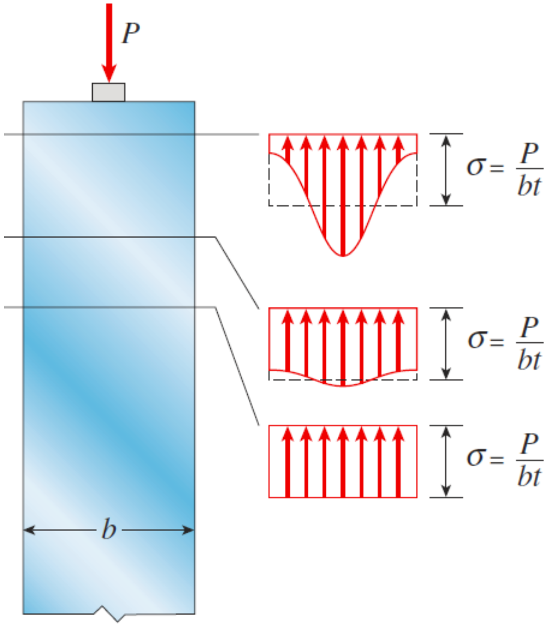
Types of Repeated Loads



Endurance Curve



Saint-Venant's Principle



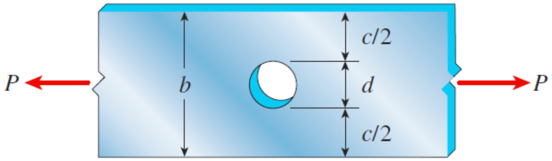
- A concentrated load produces a uniform distribution of stress within the bar at a distance equivalent to the width or diameter of the same bar.

- This principle allows one to apply standard stress formulas at cross sections a sufficient distance away from the source of the load concentration.

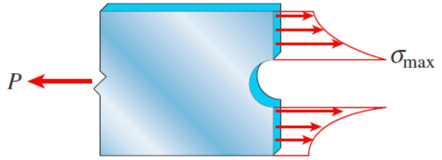
Stress Concentration Factors

- Maximum stress occurs that the edges of a hole
- May be significantly larger than the nominal stress
- Stress-Concentration Factor K
  - K factor is graphed for various geometries

$$K = \frac{\sigma_{\max}}{\sigma_{nom}}$$



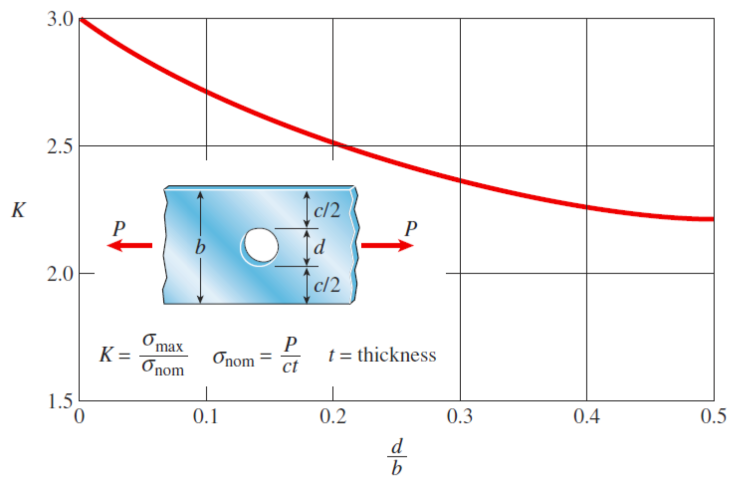
(a)



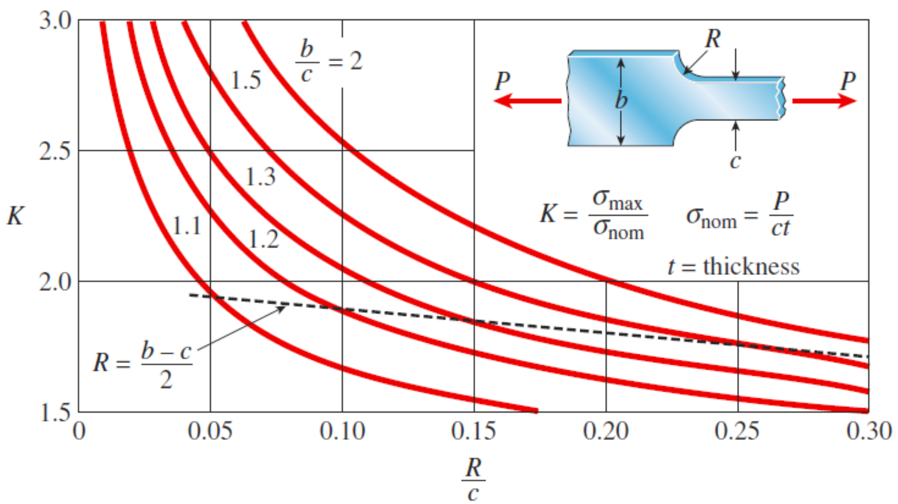
(b)

Stress Concentrations

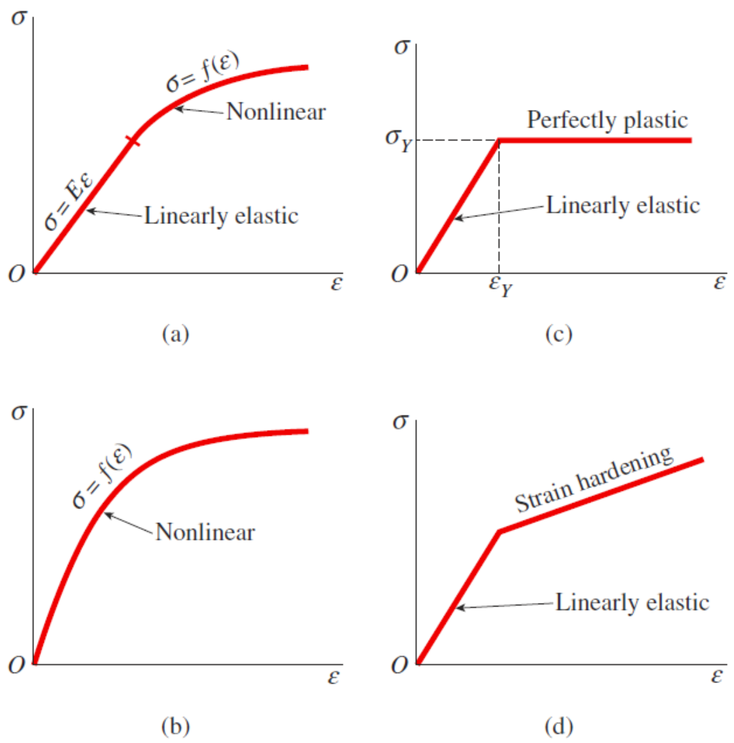
Flat Bar with Circular Holes



Flat Bar with Shoulder Fillets



Nonlinear Behavior

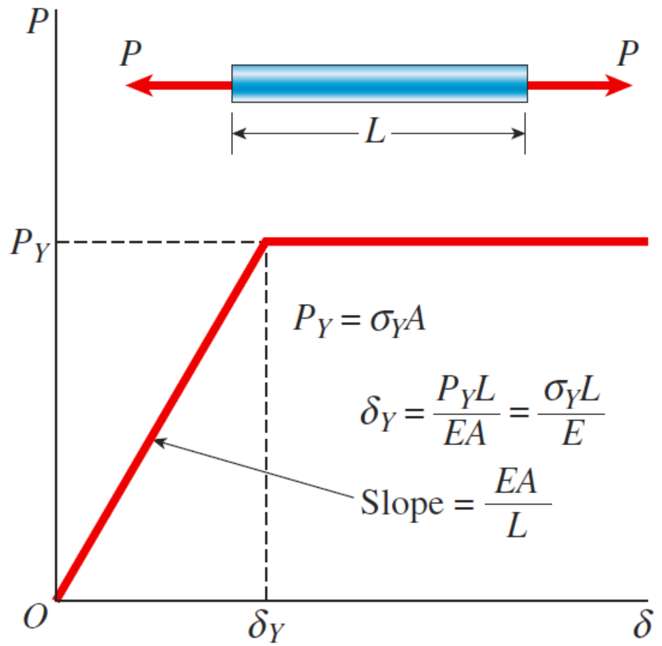


- Nonlinear behavior occurs beyond the proportional limit.

- Idealized stress-strain curves are utilized to simplify the analysis of nonlinear behavior.

- Figure A: Aluminum Approximation
- Figure B: Single Mathematical Expression
- Figure C: Structural Steel Approximation
- Figure D: Strain Hardening Approximation

Elastoplastic Analysis



- Elastoplastic material initially has a linearly elastic behavior until the onset of yielding, at which point in time the strain increases at a constant stress.
- This idealized behavior effectively approximates structural steels.
- Analysis for statically indeterminate structures is more complicated, separating the analysis into two separate: before and after yielding

Summary

- Principal Objective– Analyze axially loaded members within structures
- Force Displacement Relationship: $$\delta = \frac{PL}{EA}$$
- Stiffness: $$k = \frac{EA}{L}$$ Flexibility: $$f = \frac{L}{EA}$$
- Cables are tension-only elements and effective elastic modulus should be used.
- Segmented Members: Non-Prismatic Bars:

$$\delta = \sum_{i=1}^{n} \frac{N_i L_i}{E_i A_i}$$

$$\delta = \int_0^L d\delta = \int_0^L \frac{N(x)dx}{EA(x)}$$

Summary (cont.)

- Compatibility Equations: $\delta_{AB} = 0$ or $\delta_{AB} = gap$
- Thermal Effects: $\delta_T = \varepsilon_T L = \alpha(\Delta T)L$
- Misfits and Pre-Strains induce Pre-Stresses in statically indeterminate bars.
- Max Normal Stress: $\sigma_{\max} = \sigma_{\mathrm{x}}$ Max Shear Stress: $\tau_{\max} = \dfrac{\sigma_{\mathrm{x}}}{2}$
- Stress Concentration Factors are used to find max stresses at discontinuities.